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Possible Breakthrough: Significant Improvement of Signal to Noise Ratio by Stochastic Resonance

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Abstract. The *simplest stochastic resonator is used, a level crossing detector* (LCD), to investigate key properties of stochastic resonance (SR).

It is pointed out that successful signal processing and biological applications of SR require to work in the *large signal limit* (nonlinear transfer limit) which requires a completely new approach: *wide band input signal* and a *new, generalised definition of output noise*.

The new way of approach is illustrated by a new arrangement. The arrangement employs a special LCD, white input noise and a special, large, subthreshold wide band signal. *First time in the history of SR* (for a wide band input noise), the *signal to noise ratio becomes much higher at the output* of a stochastic resonator than *at its input*. In that way, SR is proven to have a potential to improve signal transfer. Note, that the new arrangement seems to have resemblance to *neurone models*, therefore, it has a potential also for biological applications.

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I. INTRODUCTION

In the last decade's physics literature, stochastic resonance (SR) effect has been one of the most interesting phenomena taking place in noisy non-linear systems (see e.g. [1-25]. The input of *stochastic resonator*s [12] (non-linear systems showing SR) has usually been fed by a Gaussian noise and a sinusoidal signal with frequency f_0 , that is, a random excitation and a periodic one are acting on the system. For the sake of convenience and without any restriction of generality, we shall call the measured input and output physical quantities "*voltage*" (when actual). There is an optimal strength of the input noise, where the system's output power density spectrum at the signal frequency f_0 has a maximal value. This effect is called SR. For a critical point of view about using sinusoidal signals, see Sections II and III.

From a practical point of view, the most important quantity is the "*signal to noise ratio*" (SNR) at the input (SNR_{inp}) and at the output (SNR_{out}) of the stochastic resonator. The SNR is defined as:

$$SNR = \frac{P_s}{S(f_0)} \qquad , \qquad (I.1)$$

where P_s is the mean squared value of the (background corrected) Fourier component of the input voltage at frequency f_0 , and $S(f_0)$ is the spectrum of background noise at f_0. In the case of SNR_{inp}, the background noise is itself the applied input noise voltage. In the case of SNR_{out} , the background noise is the total output voltage when no sinusoidal input signal is applied. In real SR systems, the SNR_{out} also shows a maximum, which is (in the small signal limit) located at the same strength of input noise as the maximum of P_s.

The "*old dream*" of scientists has been to find systems which can significantly increase the SNR at the output, that is,

$$SNR_{out} >> SNR_{inp} \ . \qquad (I.2)$$

There have been certain hopes [e.g. 13,14] that the principle of SR might be widely applied by nature in biological systems in order to optimise the transfer of neural signals and that SR can be used at signal processing. In this paper, we shall show that this hope ***can be*** right, however, for a real breakthrough we have to give up some old beliefs, moreover, a vast majority of previous theories and experimental results is simply irrelevant.

After the introduction (where we describe our "tools"), in Section II, we outline the new approach to succeed. In Section III, we describe the arrangement, which can realise the "old dream": the significant improvement of the SNR (see Rel. I.2.).

I/1. Do We Need a Nonlinear *Dynamical* System to Observe Stochastic

Resonance?

The answer is straightforward: ***no.*** On the contrary of the fact that SR is a *nonlinear dynamical* phenomenon, we do *not* need a dynamical system to observe it. We need only a nonlinear system with *threshold-like nonlinearity*. The key point is that the *dynamical side* of ... ıssian input

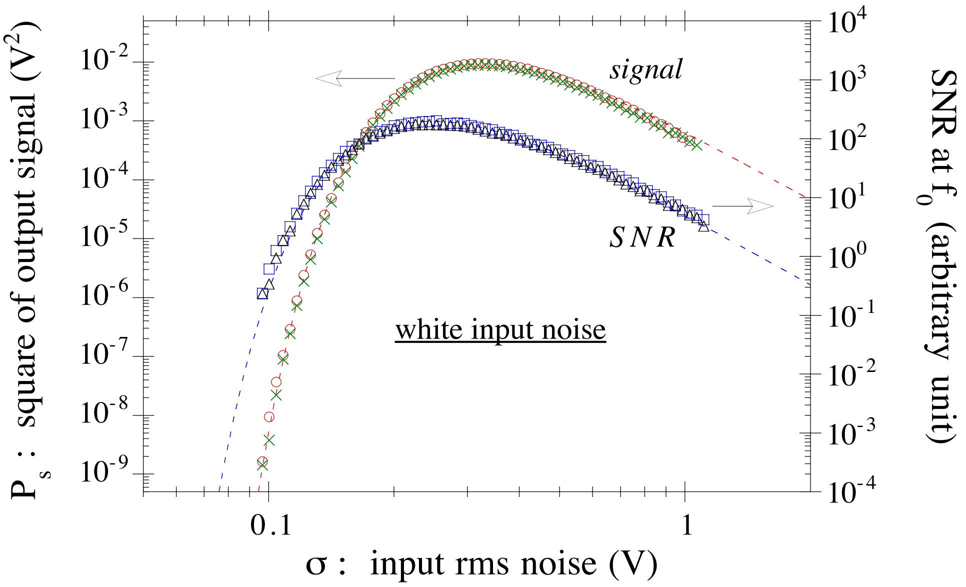

FIGURE I.1.

Representative plot of the results of white noise experiments ***in the linear response limit*** (by Gingl [15,16]) on an asymmetric LCD system. Input sinusoidal signal: B=0.1V, f_0 = 38Hz and 305Hz; input noise: white, upper cut-off 12kHz; U_t=0.45V; output pulses: A=5V and τ_0=1μs. Data: P_S: square of the output Fourier amplitude at f_0; SNR: P_S divided by the output background noise spectrum at f_0 ; (x) and (*triangle*) 305Hz; (o) and (*square*) 38Hz; curve fits: dashed lines given by the *linear* theory of Kiss (Eqs. A.3.2 and A.3.4 in the Appendix).

Until last year, it has been a common belief that stochastic resonance (SR) phenomena occur only in (bistable, sometimes monostable [10] or multistable) *dynamical* systems [1-14]. Very recently, a new stochastic resonator has been described [15,16] (invention: Moss; theory: Kiss; experimental study: Gingl): the *level-crossing detector* (LCD), which is a very practical and simple non-dynamical linear system (widely used in signal processing as an FM demodulator). Both the theory of the SR in LCD and its extensive experimental study show that the *level crossing dynamics* of the Gaussian noise *inherently* contains the SR effect. The fact that the SR curve is *not sensitive* against changing the frequency of the sinusoidal signal and the shape of the spectrum ("colour") of input noise is a strong evidence that the LCD is the simplest possible stochastic resonator, where the SR curve is purely determined by the Gaussian noise and it is not influenced by the specific dynamics of the resonator. Due to its fundamental nature, simplicity and practical importance in signal processing, we are using the LCD both for theoretical and experimental investigations. As the detailed theory of Kiss [16] has not appeared yet in the open literature, for the benefit of the reader, we present the extended theory in the Appendix.

Note, that Bulsara et al. [17], Jung [18] and Wiesenfeld et al. [19] have also studied various other aspects of threshold-crossing SR phenomena.

I/2. Description of LCD Stochastic Resonators

Asymmetric LCD stochastic resonator. The asymmetric system consists of an LCD of the following kind: whenever the instantaneous strength of the input excitation (noise and small sinusoidal signal) crosses the positive threshold level U_t *in increasing direction*, the LCD produces a positive, short pulse with amplitude A and duration τ_0 at its output. The resulting output response of the system is a random time-sequence u(t) of uniform, positive pulses.

Symmetric LCD stochastic resonator. The symmetric system consists of an LCD of the following kind: whenever the instantaneous voltage (noise and small sinusoidal signal) crosses the positive threshold level U_t *in increasing direction*, the LCD produces a positive, short pulse with amplitude A and duration τ_0 at its output; on the other hand, whenever the instantaneous strength of the input excitation (noise and sinusoidal signal) crosses the negative threshold level $-U_t$ *in decreasing direction*, the LCD produces a negative, short pulse with amplitude -A and duration τ_0 at its output. The resulting output response of the system is a random time-sequence u(t) of uniform, positive and negative pulses with zero time average.

It follows from the above definitions, that the role of a sinusoidal input signal of amplitude B and frequency f_0 can be viewed as a periodic modulation of the threshold level(s) which causes a periodic modulation of the mean frequency ν of the threshold crossings of the noise, that is the mean repetition frequency ν of pulses at the output of the system.

In Fig.1, a representative plot of the results of LCD experiments [15,16] with sinusoidal input signal and white input noise is presented and the excellent fit by the linear theory [15,16] (Eqs. A.3.2 and A.3.4 in the Appendix) is shown.

I/3. There is No Possibility to Improve the SNR in the Linear Response Limit

A corollary [20,25] given by Dykman et al. and DeWeese and Bialek, is a simple and exact proof that, in the *small signal limit* (*linear response limit, when the signal amplitude is small compared to the input noise rms amplitude*), the signal-to-noise-ratio (SNR) at the output of a stochastic resonator is *always* smaller than the SNR at the input of the resonator. The corollary is based on the fact that in the case of linear response, the input noise component belonging to an infinitesimally small bandwidth around the f_0 can also be considered as an additive input signal, due to the random-phase-oscillators representation [26] of Gaussian processes (see Eq. A.2.1. in the Appendix). Therefore, this noise component gets through the system by the same amplification as the signal does, thus, the output SNR at f_0 cannot be smaller than at the input. Moreover, if we cut out this component from the input noise, the stochastic resonator will still work and the output noise component around f_0 will be greater than zero. Due to linearity, this output noise component shows up as an excess output noise in the original case and it prohibits to reach *even* the input SNR at the output. The proof generally holds for any possible type of single stochastic resonators, for white or arbitrary coloured (Gaussian) noise. This result implies serious doubts about the applicability of those models of physical and biological systems, where the improvement of the SNR has been expected by (classical) stochastic resonance in the small signal limit.

Note that the theoretical analysis of the LCD resonator [15,16] (see Appendix) yields $SNR_{out} \leq 0.85\ SNR_{inp}$, which result is in good agreement with Gingl's simulations [15,16] and with the corollary described above.

In the scientific community of SR, the above fact empirically has been realised and this corollary explains the reason. Careful investigations on *coupled* stochastic resonator systems show, that while coupling can help to improve the SNR *compared to a single resonator*, the total gain cannot be greater than 1 , that is, the SNR at the output cannot be greater than at the input [21,22].

II. WHAT DO WE NEED FOR A BREAKTHROUGH?

We need a completely *new* way of approach. The fact that SR cannot improve the SNR in the small signal limit (the case of linear transfer) implies the need for investigations in the *strong signal limit* (the case of *nonlinear transfer*). As the research of physical, biological or other applications of SR are concerned, one has to keep in mind that the nonlinear response and its noise characteristics strongly depend on the input signal.

i) Therefore, at the experimental and theoretical study of systems with SR, the application of a *sinusoidal* input signal is *no more useful*, because a single stationary sinusoidal signal cannot transfer dynamical information. In contrast with this, an input signal with a *wide frequency* *spectrum* is desired, and its statistical properties are desired to *imitate* the properties of real physical, biological or other signals. Otherwise, the investigations do not yield a relevant conclusion for the applicability of SR.

ii) The output background noise can*not* be determined in the usual way, that is, by determining the output noise at no input signal. In the case of nonlinear transfer, the input signal yields extra cross-modulation products with the input noise, which is an extra output noise with a strong dependence on various characteristics of the input signal. Therefore, a new definition of the output noise and signal is needed. To reach this new definition, first we define the "*generalised amplification*" K(f) of the generalised (non-sinusoidal) input signal:

$$K(f) = \frac{S_{inp,out}(f)}{S_{inp}^{sig}(f)} \quad , \qquad (II.1)$$

where $S_{inp,out}(f)$ is the cross-spectrum of the input signal voltage and the *total output voltage* (output signal + output noise), and $S_{inp}^{sig}(f)$ is the (power density) spectrum of input signal. Note that K(f) depends not only on the frequency, but also on the input signal and on the input noise. The "*generalised output signal*" can be defined by its spectrum $S_{out}^{sig}(f)$:

$$S_{out}^{sig}(f) = S_{inp}^{sig}(f)\,|K(f)|^2 = \frac{|S_{inp,out}(f)|^2}{S_{inp}^{sig}(f)} . \qquad (II.2)$$

The definition of output noise spectrum $S_{out}^{noi}(f)$ is a straightforward consequence of the above definitions:

$$S_{out}^{noi}(f) = S_{out}^{tot}(f) - S_{out}^{sig}(f) \quad , \qquad (II.3)$$

where $S_{out}^{tot}(f)$ is the spectrum of the total output voltage. Note, that the above definitions restore the validity of the old definitions in the limit of small sinusoidal input signal (linear transfer and sinusoidal excitation). The new definitions work at arbitrary conditions and the only pre requirement is the stationarity of the input noise, input signal and stochastic resonator.

There the fundamental differences between the classical definition of SNR and the above described generalised quantities:

1. In the case of the *classical* definitions, the output signal is defined by the square of the frequency component of the total output voltage at the frequency of the input signal, so, that the output noise power at this frequency is subtracted.

2. In the case of the *classical* definitions, the output noise is the total output ac voltage in the case of no input signal.

3. In the case of the *generalised* quantities, the *output signal* is that part of the total output voltage which is *correlated with the input signal*.

4. In the case of the *generalised* quantities, the output noise is that part of the total output voltage which is *uncorrelated with the input signal*.

5. By comparing **2** and **4**, it is obvious that in the case of wide band signal and nonlinear response, the new definition (*correctly*) gives signal-dependent output noise and the *classical definition gives erroneously small output noise*, because of its definition with zero signal (missing cross-modulation products).

6. In the case of classical SR (linear response and sinusoidal excitation) both descriptions, the classical and the new work and *they give the same result*.

7. In the case of nonlinear response and wide band signal, only the generalised terms have realistic physical meaning at signal processing.

iii) Finally, a practical remark. Due to the nonlinearity of the transfer, the technical applications will probably have to use encoded signals (for example, pulse frequency modulation (PFM)) which are not sensitive against (non-dispersive) nonlinear transfer. Then, the necessary decoding can significantly influence the noise characteristics of the transfer, therefore this sort of investigation can also be crucial at real technical applications.

III. A POSSIBLE BREAKTHROUGH: IMPROVEMENT OF SIGNAL TO NOISE RATIO BY STOCHASTIC RESONANCE

In this section a new arrangement [23,24] is described which is able to improve the SNR by SR. In accordance with Section II, the new system is a stochastic resonator working in the strongly nonlinear transfer limit with wide band input signal. The arrangement has remarkable resemblance to neurones.

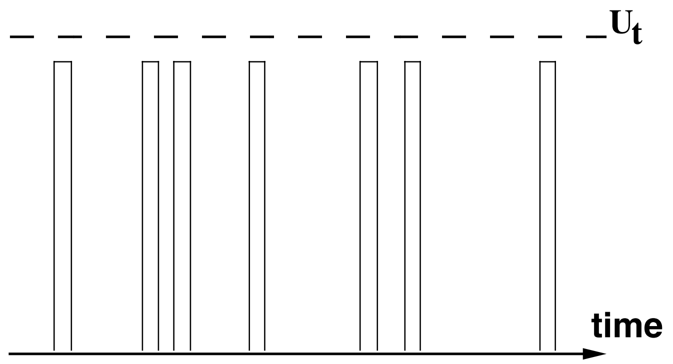

FIGURE III.1. Illustration of the wide band input signal of the new arrangement, which is similar to the input and output signal of neurones. The height, duration and mean repetition rate of pulses are A, τ_0 and ν_{sig} , respectively. The output voltage looks similar to this, except, some "jitter" of the initial times of pulses and some extra pulses at random times. The last effect corresponds to the classical output noise because it exists even without an input signal. The jitter effect yields a new sort of output noise ("jitter noise") which does not exist without input signal. It is a nice example of excess output noise described at the ii) point of Section II.

One of the simplest possible system is an asymmetric LCD, which is driven by the sum of a *Gaussian white input noise* and a *sequence of square input signal pulses*. To represent a *wide band input signal*, it is assumed that the initial times of pulses are generated by a Poissonian process with a rate ν_{sig} , however any stationary time sequence (even a periodical one) would give similar results as the improvement of the SNR is concerned. The present signal is a good representation of a real "pulse frequency modulated" (PFM) signal. As we want only to demonstrate that this system is able to *improve the SNR* significantly, for the sake of simplicity we choose the output characteristics that way, so they can make the calculations easy. Therefore, it is assumed that the square input signal pulses have the same duration τ_0 and size A *both* at the input and at the output of the LCD. Moreover, the rms amplitude σ of the input noise and A are assumed to satisfy the following relations with the threshold level U_t:

$$\sigma << U_t \quad , \qquad \text{(III.1)}$$

$$U_t - \sigma < A \qquad \text{(III.2)}$$

and the input white noise has a very high cut-off frequency f_c compared to the reciprocal length $1/\tau_0$ of pulses:

$$1/\tau_0 << f_c \quad . \qquad \text{(III.3)}$$

The quantity ν in Eq. A.2.8 (Appendix) gives the *unidirectional* threshold crossing frequencies $\nu(0)$ and $\nu(U_t\text{-}A)$ for the two possible cases: when the signal amplitude is zero and when the signal amplitude is A (that is, when the execution of a signal pulse is going on):

$$\nu(0) \approx \exp[-U_t^2/(2\sigma^2)]\ f_c/2 \qquad , \qquad \text{(III.4)}$$

$$\nu(U_t\text{-}A) \approx f_c/2 \qquad . \qquad \text{(III.5)}$$

The output voltage contains two different kinds of noises:

i) Noise due to unexpected pulses with mean repetition frequency $\nu(0)$. This noise corresponds the classical output noise in SR, because it exists even without input signal, as it is generated by the threshold crossings of input noise. In the practically interesting frequency limit ($f<<\tau_0$), Campbell-theorem applies (see Eq. A.1.2) for the spectrum $S^{noi}_{cla}(f)$ of this classical noise:

$$S^{noi}_{cla}(f) = \nu(0)\ A^2\ \tau_0^2 \qquad . \qquad \text{(III.6)}$$

ii) Noise due to the random delay ("jitter") of an output pulse when following the input pulse. This noise is an excess noise compared to the output noise of classical SR and it exists clearly due to the *joint* existence of the signal and the input noise and it is clearly related to the new way of description outlined in Sec. II. To determine the strength of "jitter noise" spectrum, the new principle is used (described in Sec. II): the output noise is that part of the input noise which is uncorrelated with the input signal. Due to Rels. III.3 and III.5, the mean duration of the jitter is very short, $\approx 2/f_c$. As the small jitter effect does not affect the bandwidth $\Delta f \approx 1/\tau_0$ of output voltage, the noise spectrum $S^{noi}_{jit}(f)$ can be approximated by $P_{jit}/\Delta f$, where P_{jit} is the mean square of that fraction of output voltage pulses which do not overlap with the input pulse due to the jitter:

$$S^{noi}_{jit}(f) \approx P_{jit}/\Delta f \approx P_{ii}\ \tau_0 \approx \tau_0\ \nu_{sig}\frac{4\ A^2}{f_c\ \tau_0}\tau_0 = \frac{4\ A^2\ \tau_0\ \nu_{sig}}{f_c} \qquad . \ \text{(III.7)}$$

The total output noise $S^{noi}_{out}(f)$ is the sum of the classical and jitter noise spectra:

$$S^{noi}_{out}(f) = S^{noi}_{cla}(f) + S^{noi}_{jit}(f) \approx \nu(0)\ A^2\ \tau_0^2 + \frac{4\ A^2\ \tau_0\ \nu_{sig}}{f_c} \qquad . \ \text{(III.8)}$$

I our case, the very small jitter (see above) provides approximately the same signal spectrum at the output as at the input (see also Eq. A.1.2):

$$S^{sig}_{out}(f) \approx S^{sig}_{in}(f) \approx \nu_{sig}\ A^2\ \tau_0^2 \qquad . \qquad \text{(III.9)}$$

As the spectrum of the input white noise is $S^{noi}_{inp} = \sigma^2/f_c$, it is now easy to calculate the ratio of SNR at the output and input by Eqs. III.8 and III.9:

$$\frac{SNR_{out}}{SNR_{inp}} \approx \frac{2\,\sigma^2}{A^2\,\tau_0^2\,f_c^2\,\exp[-(U_t/\sigma)^2/2] + 8\,A^2\,\tau_0\,\nu_{sig}} \quad , \text{(III.10)}$$

where the first term in the denominator is due to the classical output noise, and the second term is due to the jitter noise. Eq. III.10 is our mean result [23,24] which shows that the SNR can significantly be improved by this system. In Fig. III.2, the effectiveness of the arrangement is illustrated with practical parameters (see figure caption). It can be seen that with the chosen parameters, the gain in the SNR is about 10 and 100 when the repetition rate of pulses is 10Hz and 1Hz, respectively.

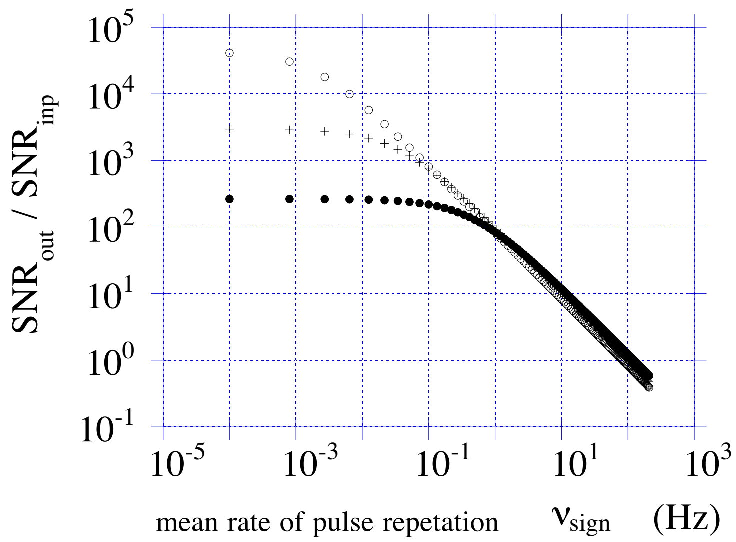

FIGURE III.2. Illustration of the effective SNR improvement of the new arrangement. Applied parameters: σ=1V; f_c=100kHz; τ_0=100μsec; A= 4.5V (filled), 5V (cross), and 5.5V (empty); U_t= 5V (filled), 5.5V (cross), and 6V (empty).

It is remarkable to note, that *close to the maximum* of possible repetition rate $\approx 1/\tau_0$, there is *no* gain, but a significant loss of SNR. As possible biological applications are concerned, the consequences of this fact has to be analysed in terms of information theory [25] and real neural signals and noise. The gain comes due to the *rare* repetition of pulses, which yields a strong reduction of the SNR at the input, but less strong SNR reduction at the output. It is due to the jitter noise component, which is proportional to the rate, similarly to the signal. At very small ν_{sig} (very rare pulses), the SNR gain is saturating due to the classical output noise which is independent of ν_{sig} . Note, the saturation limit of SNR gain can be *arbitrarily large* by choosing proper values of the A and threshold level.

Note, that Eq. III.10 is valid only in the limit given by the special conditions described by Eqs. III.1 - III.3. These conditions are chosen to prove the effectiveness of SNR improvement in a simple way. The system obviously shows SR phenomenon as a function of σ, however, the theory for arbitrarily chosen parameters is a more complicated issue.

APPENDIX: GENERALISATION OF RICE'S FORMULA FOR ZERO CROSSINGS AND THE LINEAR THEORY OF STOCHASTIC RESONANCE IN LCD SYSTEMS

At the time of completing this manuscript, the detailed Kiss-theory has not appeared yet in the open literature, therefore, for the benefit of the reader we present the theory here. The theory of asymmetric LCD system and symmetric LCD system gives the same result in the *linear limit* (sufficiently small input signal). In the non-linear case, the output signal of the symmetric system does not have even harmonics, otherwise the main features of its response is identical with the asymmetric system.

A.1 Spectra of (Weakly and Slowly) Frequency-Modulated Random Time-Sequence of Uniform Unipolar Pulses

Without frequency modulation, for sufficiently short pulses, the time-average U_{av} of the process is proportional to the mean repetition frequency ν of pulses, $U_{av} = \langle u(t) \rangle_t = \nu A \tau_0$. The small and slow modulation of the ν yields the linear modulation of the time-average U^*_{av} measured for short times (e.g. much shorter than the modulation frequency), so:

$$U^*_{av}(t) = \nu(t)\, A\tau_0 \quad , \qquad (A.1.1)$$

which, in the case of a sinusoidal modulation, results in a sinusoidal component in the Fourier-spectrum of the pulse train. Moreover, this component has zero phase-shift and its amplitude is *independent* from the modulation frequency. The validity of this result requires to fulfil the following practical precondition:

(i) The modulation frequency $f_0 << \nu(t)$. This requirement is also necessary for the efficient transfer of modulating signal.

Under the condition i), Eq. A.1.1 describes the periodic response for the modulation in the pulse-train. However, it does not give information on the spectrum of the noise-component of the pulse-train. For pulses with completely random initial times, in the low frequency limit of $0 < f << \tau_0^{-1}$, the Campbell-theorem describes the spectrum $S_{no}(f)$ for the case of no frequency-modulation:

$$S_{no}(f) = \nu A^2 \tau_0^2 \quad . \qquad (A.1.2)$$

Assuming, that the modulation of the mean repetition frequency of the pulses is symmetric, weak and small, we can expect Eq. A.1.2 correctly describes the background noise spectrum of the pulse-train. In Eq. A.1.2 , the values of ν for no frequency modulation applies.

It is obvious from Eqs. A.1.1 and A.2.2, that the quantity $\nu(t)$ has a crucial role in the determination of the Fourier-components of the pulse-train, so we devote the next section to the derivation of $\nu(t)$.

A.2. Generalisation of the Rice-Formula of Zero Crossings for the Case of Arbitrary Level Crossings

Rice [26] determined the mean frequency ν_0 of zero-crossings of the amplitude of Gaussian noises. In order to obtain the mean frequency $\nu(U_t)$ of crossings of arbitrary levels U_t we derive the Rice-formula in a different and simpler way which yields the required generalisation. We are concerned with physical noises which means that the amplitude y(t) of the Gaussian noise and its velocity dy/dt have finite root-mean-square (rms) values σ and Δ , respectively. Thus, their power density spectrum have cut-offs (at least for high frequencies).

First, it is shown that the amplitude y(t) of the Gaussian noise is statistically independent of its velocity dy/dt. According to the random-phase-oscillators representation [26] of Gaussian processes, the amplitude of a Gaussian noise can be written as:

$$y(t)=\sum_{n=1}^{\infty} a_n \sin(2\pi f_n t+\phi_n) \quad , \quad \text{(A.2.1)}$$

where a_n , f_n and ϕ_n are the amplitude, the frequency and the random initial phase of the n-th oscillator, respectively. The oscillator frequency is given as $f_n= n\,\delta f$, where δf is infinitesimally small. The random set of ϕ_n (uniformly distributed in the range of $(0,2\pi]$) is different for each representation of the given noise process. From Eq. A.2.1, the velocity can be written, as:

$$\frac{dy(t)}{dt}=\sum_{n=1}^{\infty} 2\pi f_n a_n \cos(2\pi f_n t+\phi_n) \quad . \quad \text{(A.2.2))}$$

It is easy to see that the cross-correlation function of Eqs. A.2.1 and A.2.2 is zero due to the orthogonality of sinus and cosine functions with the same frequency and due to the orthogonality of sinusoidal functions with different frequencies:

$$\langle y(t)\, dy(t)/dt \rangle_t = 0 \quad , \quad \text{(A.2.3)}$$

where the index t represents time-averaging (as y(t) and dy/dt are stationary and ergodic processes). Using the well known fact that a zero cross-correlation of two different Gaussian processes implies their statistical independence, we can conclude that the *instantaneous amplitude and the instantaneous velocity of a Gaussian noise are statistically independent from each other*.

In the next step, we determine the functional form of the mean frequency $\nu_0(U_t)$ of the crossings of level U_t by the noise amplitude y(t). Let us consider the behaviour of the noise in an infinitesimally narrow amplitude interval U_t-∂U ≤ y(t) ≤ U_t-∂U around U_t. The infinitesimal smallness of the interval 2∂U has two important implications: i) the amplitude distribution function of the noise will be uniform in this range; ii) for a long but finite duration of the noise, the number of those amplitude trajectories which enter into the interval and, after changing their direction, leave the interval without crossing the level U_t , is zero. Note, the last implication pre-requires the frequency band limited property of the physical Gaussian noise (see above). On the other hand, the number of amplitude trajectories which enter into the interval and, without changing their direction, leave the interval via crossing the level U_t , is not zero and it is related to the mean frequency $\nu(U_t)$ of crossing this level. On the base of these considerations the following relations can be written for the probability of finding the instantaneous amplitude y(t) within the interval:

$$2\, g(U_t)\, \partial U = \nu(U_t)\, \partial t \qquad , \qquad (A.2.4)$$

where g(U) is the amplitude distribution of the noise, that is, $g(U_t) = (2\pi)^{-1/2}\, \sigma^{-1} \exp[-(U_t/\sigma)^2/2]$ and ∂t is the mean passing time of trajectories via the interval. The right hand side of Eq. A.2.4 represents the fraction of time which the noise amplitude spends in the interval. From Eq. A.2.4, the level crossing frequency can be given as follows:

$$\nu(U_t) = 2\, \frac{\partial U}{\partial t}\, (2\pi)^{-1/2}\, \sigma^{-1} \exp[-(U_t/\sigma)^2/2] \qquad (A.2.5)$$

The quantity 2∂U/∂t is equal to the mean velocity of the noise in the interval. Due to the statistical independence of the velocity and the amplitude, ∂U/∂t will be independent from the location of the interval, so the value of U_t. On the other hand, ∂U/∂t is proportional to the rms velocity of the noise, consequently:

$$\nu(U_t) = c\, \Delta\, \sigma^{-1} \exp[-(U_t/\sigma)^2/2] \qquad , \qquad (A.2.6)$$

where c is a constant. With the help of the power-density spectrum S(f) of the noise y(t) , the Δ term can be replaced by the integral of $(2\pi)^2\, f^2\, S(f)$:

$$\nu(U_t) = C\, \sigma^{-1} \exp\left[\frac{-U_t^2}{2\sigma^2}\right] \left[\int_0^{\infty} f^2\, S(f)\, df\right]^{1/2} \quad , \quad (A.2.7)$$

where $C=4\pi^2 c$. Eqs. A.2.6 and A.2.7 describe the functional form of the dependence of ν on S(f) and U_t . The determination of the constants c and C is based on the following trick: we take U_t=0 as threshold (so the value of the exponent term becomes 1) and a narrow-band noise with the following spectrum: $S(f)=S_0$ for $f_0-\partial f < f < f_0+\partial f$ and otherwise it is zero.

Then, in the limit $\partial f \rightarrow 0$, Eq. A.2.7 will approach the zero-crossing frequency of a sinusoidal signal with frequency f_0 , that is, $\nu = 2f_0$. In this way we have obtained our final formula:

$$\nu(U_t) = 2\,\sigma^{-1} \exp\left[\frac{-U_t^2}{2\sigma^2}\right] \left[\int_0^{\infty} f^2\, S(f)\, df\right]^{1/2} , \qquad \text{(A.2.8)}$$

which is the generalisation of the Rice-formula [26] (derived for the frequency of zero-crossings). Indeed, in the limit of U_t=0 , Eq. A.2.8 becomes identical with the formula derived by Rice [26].

A.3 Stochastic Resonance in Level Crossing Detectors: Theory for the Case of Small Sinusoidal Signal (Linear Signal Transfer)

When the pulse-sequence is generated in an LCD system (of threshold U_t) by the sum of a Gaussian noise and a sinusoidal signal $B\sin(\omega t)$, we can view the problem as the level-crossing problem of the pure noise with a modulation $-B\sin(\omega t)$ of the threshold level U_t . To describe the quantity $\nu(t)$ for this case, one can apply Eq. A.2.8 provided the following preconditions holds: condition i) and (ii). The modulation frequency $f_0 << \tau_{corr}$, (where τ_{corr} is the correlation time of the input noise). In the case of noises with a flat spectrum, the validity the condition i) (Section A.1.1) implies the validity of condition ii), however, for other noises (like band-limited 1/f noise) it does not necessarily imply that.

Then, condition i) and ii) provide that the pulse-sequence will represent its real statistical properties during the half-period $1/2f_0$ of the sinusoidal modulation, that is, Eq. A.2.8 does have a real meaning. Taking into the account that the ν at one-directional level crossing is half of the ν at bi-directional level crossing, the time-dependence $\nu(t)$ of the mean repetition frequency of output pulses in an asymmetric LCD system can be given as:

$$\nu(t) = \sigma^{-1} \exp\left\{\frac{-\,[U_t - B\sin(\omega t)]^2}{2\sigma^2}\right\} \left[\int_0^{\infty} f^2\, S(f)\, df\right]^{1/2} . \qquad \text{(A.3.1)}$$

From Eqs. A.1.1 and A.3.1, by Taylor-expanding the exponential function and keeping the linear term, we obtain the squared amplitude P_s of the output sinusoidal response at frequency f_0 :

$$P_s = (\nu_0\, B\, A\, \tau_0\, U_t)^2\, \sigma^{-4} \exp\left[-(U_t/\sigma)^2\right] , \qquad \text{(A.3.2)}$$

where the phase-shift of the first harmonic is zero compared to the input sinusoidal signal and ν_0 is the mean frequency of one-directional zero-crossings of the input noise and it can be given from the original Rice-formula [26] (see Eq. A.2.8):

$$\nu_0 = 2\ \sigma^{-1} \left[\int_0^{\infty} f^2\, S(f)\, df \right]^{1/2} . \qquad (A.3.3)$$

In order to satisfy the preconditions of the applicability of Campbell-theorem, that is, for Eq. A.1.2, we have to apply a new precondition:

iii) Let us assume that $\nu(t) << \tau_{corr}^{-1}$.

This condition provides random initial times of the pulses. Then, from Eq. A.1.2, Eq. A.3.1 with B=0 and Eq. A.3.2 we obtain the SNR at the output:

$$SNR_{out} = \nu_0\, (B\, U_t)^2\, \sigma^{-4} \exp[-(U_t/\sigma)^2/2] \quad , \qquad (A.3.4)$$

where $SNR_{out} = \frac{P_s}{S_{no}(f_0)}$ is the ratio of P_s and the background noise spectrum at f_0 . An upper limit for the ratio of SNR at the output and at the input can be deduced:

$$\frac{SNR_{out}}{SNR_{inp}} \le \frac{4}{e\sqrt{3}} \approx 0.85 \qquad (A.3.5)$$

which is in agreement with experimental results and with Dykman's corollary [20] about the inapplicability of linear SR for the improvement of SNR.

A.4 Analysis of the Theoretical Results

Analysing Eqs. A.3.2 and A.3.4, it turns out that in the limit of our approximation, the *only important properties* of the input noise are its rms amplitude σ and its rms velocity, the particular structure of its spectrum S(f) does not have influence on the strength of output signal. Moreover, the frequency of the input signal does *not* play any role in Eqs. A.3.2 and A.3.4. It is important to note that these fundamental properties are present only in this threshold system, even the very recent threshold SR systems proposed by Jung [18] and Bulsara [19] do have a frequency dependent SR behaviour.

Note that, when the preconditions of the outlined theory were *not* fulfilled, we have naturally found deviations from Eqs. A.3.2 and A.3.4. For example, at very small input noise, the linear approach breaks down; and at very large input noise, correlations between level-crossing times can cause deviations from the linear SNR curve. The strongly nonlinear LCD arrangement described in Section III behaves in a completely different way.

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